

Approximation in law to operator fractional Brownian motion

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Abstract

As a well-known extension of the famous fractional Brownian motion, the operator fractional Brownian motion has been studied extensively. One direction of these studies is to study weak limit theorems for this kind of processes. In this paper, we also go along with this direction. We show that the operator fractional Brownian motion can be approximated in law by some non-linear functions of stationary Gaussian vector-valued variables.

Keywords: Operator fractional Brownian motion, Hermite polynomial, weak convergence.

1. Introduction

Recall that we call some processes self-similar processes if they are invariant in distribution under suitable scaling of time and space. This kind of processes are first introduced rigorously by Lamperti [12] under the name “semi-stable”. Since then, it has attracted a lot of interest. We refer to Vervaat [22] for general properties, to Samorodnitsky and Taqqu [18] [Chaps.7 and 8] for studies on Gaussian and stable self-similar processes and random fields. Scholars have extended the definition of self-similarity to allow for scaling by linear operators on \mathbb{R}^d . Let $End(\mathbb{R}^d)$ be the set of linear operators on \mathbb{R}^d (endomorphisms) and $Aut(\mathbb{R}^d)$ be the set of invertible linear operators (automorphisms) in $End(\mathbb{R}^d)$. For convenience, we will not distinguish an operator $D \in End(\mathbb{R}^d)$ from its associated matrix relative to the standard basis of \mathbb{R}^d . Recall that an \mathbb{R}^d -valued stochastic process $\tilde{Y} = \{\tilde{Y}(t), t \in \mathbb{R}\}$ is said to be operator self-similar (o.s.s.) if it is stochastically continuous, and there exists a $D \in End\{\mathbb{R}^d\}$ such that for every $c > 0$

$$\tilde{Y}(ct) \stackrel{d}{=} c^D \tilde{Y}(t), \quad t \in \mathbb{R}, \quad (1.1)$$

where $\stackrel{d}{=}$ denotes equality of all finite-dimensional distributions, and

$$c^D = \exp((\log c)D) = \sum_{k=0}^{\infty} \frac{1}{k!} (\log c)^k D^k.$$

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Moreover, we call D in (1.1) the *exponent* of the o.s.s process \tilde{Y} . For more information on this kind of processes, refer to Laha and Rohatgi [13], Hudson and Mason [11], and Sato [17].

Corresponding to the fractional Brownian motion (FBM) in one-dimensional case ($d = 1$), there exists an operator fractional Brownian motion (OFBM) in multidimensional case ($d \geq 2$). OFBMs are mean-zero, o.s.s., Gaussian processes with stationary increments. They are of interest in several areas for similar reasons to those in the univariate case. For example, see Chung [2], Davidson and de Jong [4], Didier and Pipiras [8, 9] and the references therein.

Weak convergence to FBMs has been studied extensively since the works of Davydov [3] and Taqqu [21]. As an extension of FBMs, some results on approximations of OFBMs have also been established. For example, Marinucci and Robinson [15] presented a weak limit theorem for a special class of OFBMs via a sequence of random variables. Dai [5] shown that a special kind of OFBMs can be approximated in law by a stationary sequence of vector-valued Gaussian random variables. We should point out that they all studied some special kinds of OFBMs, but for the general ones introduced by Didier and Pipiras [8], as far as we know, there is no work studying their weak limit theorems. Inspired by this, we present a weak limit theorem for the general OFBM in this work.

The asymptotical distribution of non-linear functionals of Gaussian fields has been extensively studied. See, for example, Arcones [1] and Sánchez [19, 20]. In particular, Taqqu [21] showed that the FBM can be approximated in law by a sequence of non-linear functions of Gaussian random variables. In this paper, we proceed with Taqqu [21], and show that the OFBM can also be approximated by non-linear functions of vector-valued Gaussian random variables.

The rest of this paper is organized as follows. Section 2 is devoted to discussing weak convergence of vector-valued stationary processes. In Section 3, we discuss the weak limit theorems for non-linear functions of Vector-valued Gaussian random variables. In Section 4, the main result of this paper is presented. A final note at the end of this paper is devoted to discussing the possibility of generalization of the methods and results appearing in this paper.

Most of the estimates of this paper contain unspecified constants. An unspecified positive and finite constant will be denoted by \tilde{K} , which may not be the same in each occurrence. Sometimes we shall emphasize the dependence of these constants upon parameters.

2. Conditions for Weak Convergence

Let $\{Z_N(t), t \in [0, 1]\}_{N \in \mathbb{N}}$ be a sequence of vector-valued functions. In this section, we discuss the possibility of weak convergence of this sequence. Before we state the main result of this section, we recall some basic knowledges. Throughout this paper, let B^* be the adjoint operator of B . We use $\|x\|_2$ to denote the usual Euclidean norm of $x = (x^1, \dots, x^d)^T \in \mathbb{R}^d$, where x^T denotes the transpose of a vector x in \mathbb{R}^d . Moreover, let $\|A\| = \max_{\|x\|_2=1} \|Ax\|_2$ denote the operator norm of A . It is well-known that for $A, B \in \text{End}(\mathbb{R}^d)$,

$$\|AB\| \leq \|A\| \cdot \|B\|, \quad (2.1)$$

and for every $A = (A_{i,j})_{d \times d} \in \text{End}(\mathbb{R}^d)$,

$$\max_{1 \leq i,j \leq d} |A_{i,j}| \leq \|A\| \leq d^{\frac{3}{2}} \max_{1 \leq i,j \leq d} |A_{i,j}|. \quad (2.2)$$

Furthermore, let

$$\lambda_A = \min\{\text{Re}\lambda : \lambda \in \sigma(A)\} \text{ and } \Lambda_A = \max\{\text{Re}\lambda : \lambda \in \sigma(A)\}, \quad (2.3)$$

where $\sigma(A)$ is the collection of all eigenvalues of A .

In order to simplify our discussion, we need the following notation. Let $A(n) = (A_{ij}(n))_{d \times d} \in \text{End}\{\mathbb{R}^d\}$ and $B(n) = (B_{ij}(n))_{d \times d} \in \text{End}\{\mathbb{R}^d\}$, we say that $\{A(n)\}$ is asymptotically equivalent to $\{B(n)\}$, as $n \rightarrow \infty$, if $A_{ij}(n)/B_{ij}(n) \rightarrow 1$, as $n \rightarrow \infty$, for all i and j . We denote this by $A(n) \sim B(n)$, as $n \rightarrow \infty$.

Let $A = (A_{i,j})_{d \times d} \in \text{End}\{\mathbb{R}^d\}$ and $B = (B_{i,j})_{d \times d} \in \text{End}\{\mathbb{R}^d\}$. If $\sum_{i,j=1}^d |A_{i,j}| \leq \sum_{i,j=1}^d |B_{i,j}|$, then we say $A \leq B$. For the relation $A \leq B$, we have the following two properties.

Lemma 2.1 *If $A \leq B$, then $\|A\| \leq K\|B\|$.*

Proof: For any $A = (A_{i,j}) \in \text{End}\{\mathbb{R}^d\}$, we define

$$\|A\|_3 = \sum_{i,j=1}^d |A_{i,j}|.$$

One can easily verify that $\|A\|_3$ is a norm. Since all the norms in a finite-dimensional space are equivalent, we easily see that lemma holds. \square

Lemma 2.2 *If $A \leq B$, then for all $D \in \text{End}\{\mathbb{R}^d\}$ and $\tilde{K} \in \mathbb{R}_+$,*

$$AD \leq \tilde{K}BD, \text{ and } DA \leq \tilde{K}DB. \quad (2.4)$$

Lemma 2.2 can be easily proved, so we skip the proof. We next introduce some technical lemmas, which play an important role in this section. The following lemma can be found in Mason and Xiao [16].

Lemma 2.3 *Let $D \in \text{End}(\mathbb{R}^d)$. If $\lambda_D > 0$ and $r > 0$, then for any $\delta > 0$, there exist positive constants K_1 and K_2 such that*

$$\|r^D\| \leq \begin{cases} K_1 r^{\lambda_D - \delta}, & \text{for all } r \leq 1, \\ K_2 r^{\lambda_D + \delta}, & \text{for all } r \geq 1. \end{cases} \quad (2.5)$$

In order to prove weak convergence, we need the following tightness criterion in the space $\mathcal{D}^d([0, 1]) = \mathcal{D}^d([0, 1], \mathbb{R}^d)$.

Lemma 2.4 *Let $\{Z_n(t)\}_{n \in \mathbb{N}}$ be a sequence of stochastic processes in $\mathcal{D}^d([0, 1])$ satisfying:*

(i) *for every $n \in \mathbb{N}$, $Z_n(0) = 0$ a.s.;*

(ii) there exist constants $K > 0$, $\beta > 0$, $\alpha > 1$ and an integer $N_0 \in \mathbb{N}$ such that

$$\mathbb{E} \left[\left\| Z_n(t) - Z_n(s) \right\|_2^\beta \right] \leq \tilde{K} (t-s)^\alpha, n \geq N_0 \text{ and } 0 \leq s \leq t \leq 1. \quad (2.6)$$

Then $\{Z_n(t)\}$ is tight in $\mathcal{D}^d([0, 1])$.

This result is classical, so we omit the proof.

Before we state the main result of this section, we introduce the following notation. Let $\{Y_i\}_{i \in \mathbb{N}}$ be a stationary mean-zero sequence of vector-valued random variables with $\mathbb{E}[\|Y_i\|_2^2] < \infty$. For any $N \in \mathbb{N}$, define

$$S_N(t) = \sum_{i=1}^{\lfloor Nt \rfloor} Y_i,$$

where $\lfloor x \rfloor$ denotes the greatest integer not more than x . For convenience, let $S_N = S_N(1)$. Furthermore, we adapt the following convention.

Convention: Empty sums are equal to $(0, \dots, 0)^T$.

The main result of this section is the following.

Theorem 2.1 Suppose that the sequence $\{Z_N(t)\}_{N \in \mathbb{N}}$ of random functions in $\mathcal{D}^d([0, 1])$ satisfies:

(i)

$$Z_N(t) = N^{-D} S_N(t) \quad (2.7)$$

with $0 < \lambda_D, \Lambda_D < 1$ and $D \in \text{End}(\mathbb{R}^d)$.

(ii) As $N \rightarrow \infty$,

$$\mathbb{E}[S_N S_N^T] \sim N^D \Gamma N^{D*}, \quad (2.8)$$

where Γ is a $d \times d$ symmetric positive semi-definite matrix.

(iii) As $N \rightarrow \infty$,

$$\mathbb{E}[\|S_N\|_2^{2\alpha}] \sim \left\| \mathbb{E}[S_N S_N^T] \right\|^\alpha \quad (2.9)$$

with $\alpha > \frac{1}{2\lambda_D}$.

(iv) The finite-dimensional distributions of $Z_N(t)$ converge as $N \rightarrow \infty$.

Then the sequence $\{Z_N(t)\}$ converges weakly, as $N \rightarrow \infty$, to an operator self-similar processes $X = \{X(t)\}$ with o.s.s exponent D and stationary increments, whose finite-dimensional distributions are the limits of those of $\{Z_N(t)\}$.

Proof: We choose $0 \leq s \leq t \leq 1$. In order to prove Theorem 2.1, we first prove that $\{Z_N(t)\}$ is tight. In fact, we have

$$\mathbb{E} \left[\|Z_N(t) - Z_N(s)\|_2^{2\alpha} \right] = \mathbb{E} \left[\|Z_{\lfloor Nt \rfloor - \lfloor Ns \rfloor}\|_2^{2\alpha} \right], \quad (2.10)$$

since $\{Y_i\}_{i \in \mathbb{N}}$ is stationary. Hence it follows from (2.1), (2.7) and Lemma 2.3 that for any $\delta > 0$ with $\alpha(\lambda_D - \delta) > \frac{1}{2}$,

$$\mathbb{E} \left[\|Z_N(t) - Z_N(s)\|_2^{2\alpha} \right] \leq \tilde{K} \frac{1}{N^{2\alpha(\lambda_D - \delta)}} \mathbb{E} \left[\|S_{\lfloor Nt \rfloor - \lfloor Ns \rfloor}\|_2^{2\alpha} \right]. \quad (2.11)$$

On the other hand, we should note that for any $X \in \mathbb{R}^d$

$$\|X\|_2^2 = \sum_{k=1}^d (x^k)^2. \quad (2.12)$$

Hence, it follows from (2.9) and (2.12) that there exists $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$

$$\mathbb{E} \left[\|S_{\lfloor Nt \rfloor - \lfloor Ns \rfloor}\|_2^{2\alpha} \right] \leq \tilde{K} \left\| \mathbb{E} \left[S_{\lfloor Nt \rfloor - \lfloor Ns \rfloor} S_{\lfloor Nt \rfloor - \lfloor Ns \rfloor}^T \right] \right\|^\alpha. \quad (2.13)$$

Noting that $\{Y_i\}_{i \in \mathbb{N}}$ is stationary, it follows from (2.8) and (2.13) that

$$\mathbb{E} \left[\|S_{\lfloor Nt \rfloor - \lfloor Ns \rfloor}\|_2^{2\alpha} \right] \leq \tilde{K} \left\| \left[\frac{\lfloor Nt \rfloor - \lfloor Ns \rfloor}{N} \right]^D N^D \Gamma N^{D*} \left[\frac{\lfloor Nt \rfloor - \lfloor Ns \rfloor}{N} \right]^{D*} \right\|^\alpha. \quad (2.14)$$

Hence, it follows from (2.11), (2.14) and Lemma 2.3 that

$$\mathbb{E} \left[\|Z_N(t) - Z_N(s)\|_2^{2\alpha} \right] \leq \tilde{K} \left[\frac{\lfloor Nt \rfloor - \lfloor Ns \rfloor}{N} \right]^{2(\lambda_D - \delta)\alpha}, \quad (2.15)$$

since $t, s \in [0, 1]$.

On the other hand, due to de Haan [7], we have

$$\lim_{N \rightarrow \infty} \left[\frac{\lfloor Nt \rfloor - \lfloor Ns \rfloor}{N} \right]^{2(\lambda_D - \delta)\alpha} = (t - s)^{2(\lambda_D - \delta)\alpha} \quad (2.16)$$

holds uniformly for $t, s \in [0, 1]$. Hence, it follows from (2.11), (2.14) and (2.16) that, for any $\delta > 0$, there exists a constant $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$

$$\mathbb{E} \left[\|Z_N(t) - Z_N(s)\|_2^{2\alpha} \right] \leq \tilde{K} (t - s)^{2\alpha(\lambda_D - \delta)}. \quad (2.17)$$

Finally, it follows from Lemma 2.4 and (2.17) that $\{Z_N(t)\}$ is tight.

The tightness and convergence of the finite-dimensional distributions ((iv) of Theorem 2.1) ensure the weak convergence of $Z_N(t)$ to some limiting process $X(t)$. Since $\{Y_i\}$ is stationary, $X(t)$ must have stationary increments.

Next, we show that $\{X(t)\}$ is continuous. From the fact that the finite-dimensional distributions of $Z_N(t)$ converge to those of $X(t)$, we can get there exists $N_1 \in \mathbb{N}$ such that

$$\mathbb{E} \left[\|X(t) - X(s)\|_2^{2\alpha} \right] = \lim_{N \rightarrow \infty} \mathbb{E} \left[\|Z_N(t) - Z_N(s)\|_2^{2\alpha} \right] \quad (2.18)$$

$$\leq \tilde{K} (t - s)^{2\alpha(\lambda_D - \delta)}. \quad (2.19)$$

It follows from Proposition 10.3 in Ethier and Kurtz [10] that $X(t)$ has a.s. continuous modification $\{\bar{X}(t)\}$. Finally, from the above arguments, we get from Hudson and Mason [11] that $X(t)$ is operator self-similar. \square

Remark 2.1 From the proof of Theorem 2.1, we have that the condition (ii) in Theorem 2.1 can be replaced by the following conditions (ii') and (iii'), respectively. For $N \in \mathbb{N}$ large enough,

$$(ii') \quad \left\| \mathbb{E} \left[S_N S_N^T \right] \right\| \leq \tilde{K} \left\| N^D \Gamma N^{D*} \right\|, \quad (2.20)$$

and

$$(iii') \quad \mathbb{E} \left[\|S_N S_N^T\|_2^{2\alpha} \right] \leq \tilde{K} \left\| \mathbb{E} [S_N S_N^T] \right\|^\alpha. \quad (2.21)$$

Remark 2.2 (1) If the limiting process $X(t)$ is a Gaussian process, we can easily get that $X(t)$ is a time reversible process.

(2) The matrix Γ is the covariance matrix of the limiting process $X(1)$.

3. Limit Theorems for Non-linear Functions

In this section, we mainly discuss the limit theorems for non-linear functionals of stationary vector-valued Gaussian sequence. In the rest of this paper, in order to simplify our discussion, we only consider this problem in the two-dimensional space.

3.1. Conditions for Weak Convergence

Let $\{X_i = (X_i^1, X_i^2)^T\}$ be a stationary vector-valued Gaussian sequence with zero-mean. Let $r(i, j) = r(|i - j|) = \mathbb{E}[X_i X_j^T] = (r_{p,q}(i, j))_{2 \times 2}$ be its correlation function.

Below, we discuss that what conditions can be imposed on a function G and on the sequence $r(n)$ such that $\sum_{i=1}^{[Nt]} G(X_i)$ converges weakly to a process, as $N \rightarrow \infty$.

Let

$$H_l(x) = (-1)^l e^{\frac{x^2}{2}} \frac{d^l}{dx^l} e^{-\frac{x^2}{2}}, \quad l \in \mathbb{N}$$

be the Hermite polynomials. In order to introduce the Hermite rank of a function, we need the following notation. Inspired by Sánchez [19], let

$$e_L(X) = (H_{l_1}(X^1) H_{l_2}(X^2), 0)^T,$$

and

$$\tilde{e}_L(X) = (0, H_{l_1}(X^1) H_{l_2}(X^2))^T,$$

where $X = (X^1, X^2)^T$ and $L = (l_1, l_2)^T$. Furthermore, let $G : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a measurable vector-valued function. Inspired by Arcones [1], Sánchez [19] and Taqqu [21], we define the Hermite rank of a function $G(X)$.

Definition 3.1 If $\mathbb{E}[\|G\|_2^2] < \infty$, and $G(X)$ has zero-mean, then we define the *Hermite rank* of G as follows.

$$\text{Rank}(G) = \inf \left\{ \tau : \exists L = (l_1, l_2)^T \text{ with } l_1 + l_2 = \tau, \text{ such that} \right. \quad (3.1)$$

$$\left. \mathbb{E}[G^T(X)e_L(X)] \neq 0 \text{ or } \mathbb{E}[G^T(X)\tilde{e}_L(X)] \neq 0 \right\}. \quad (3.2)$$

Moreover, define:

$$\mathbb{G}_m = \{G : \mathbb{E}[G(X)] = (0, 0)^T, \mathbb{E}[\|G(X)\|_2^2] < \infty, \text{ and } \text{Rank}(G) = m\}.$$

In order to answer the problem mentioned at the beginning of this subsection, we assume the following condition.

Definition 3.2 Let $D \in \text{End}\{\mathbb{R}^2\}$ with $\frac{1}{2} < \lambda_D, \Lambda_D < 1$. We say that a stationary sequence $\{X_i\}$ of vector-valued Gaussian random variables with zero-mean satisfies **Condition** $\mathcal{H}(m, D)$, if

(i) for $N \in \mathbb{N}$ large enough,

$$\sum_{i=1}^N \sum_{j=1}^N \left\| (r(i, j)) \right\|^m \leq \tilde{K} \|N^D \Gamma N^{D*}\|, \quad (3.3)$$

where Γ is a $d \times d$ symmetric positive semi-definite matrix;

(ii) as $n \rightarrow \infty$,

$$\left\| (r_{p,q}(|n|)) \right\| \rightarrow 0. \quad (3.4)$$

The following theorem answers the question we mentioned at the beginning of this section.

Theorem 3.1 If $G \in \mathbb{G}_m$ and $\{X_i\}$ satisfies **Condition** $\mathcal{H}(m, D)$, then for large enough $N \in \mathbb{N}$,

$$\left\| \mathbb{E} \left[\left(\sum_{i=1}^N G(X_i) \right) \left(\sum_{i=1}^N G^T(X_i) \right) \right] \right\| \leq \tilde{K} \|N^D \Gamma N^{D*}\|. \quad (3.5)$$

Proof of Theorem 3.1: Inspired by Major [14] and Sánchez [19], we can expand $G(X_i)$ as

$$G(X_i) = \sum_{r=0}^{\infty} \sum_{L \in I_r} \{C_L e_L(X_i) + \tilde{C}_L \tilde{e}_L(X_i)\}, \quad (3.6)$$

where $L = (l_1, l_2)^T$, $I_r = \{l_1 + l_2 = r\}$, and $C_G = \sum_{r=0}^{\infty} \sum_{L \in I_r} \{C_L^2 + \tilde{C}_L^2\} l_1! l_2! < \infty$.

It follows from Sánchez [19] that for any $i, j \in \mathbb{N}$,

$$\begin{aligned} & \mathbb{E}[G(X_i)G^T(X_j)] \\ &= \sum_{r=0}^{\infty} \sum_{K, L \in I_r} \left\{ C_L C_K \mathbb{E}[e_L(X_i) e_K^T(X_j)] + \tilde{C}_L \tilde{C}_K \mathbb{E}[\tilde{e}_L(X_i) \tilde{e}_K^T(X_j)] \right\}, \end{aligned} \quad (3.7)$$

since $e_L(X_i)\tilde{e}_L^T(X_j) = 0$. Since the rank of G is m , we can rewrite the equation (3.7) as follows.

$$\begin{aligned}
& \mathbb{E}[G(X_i)G^T(X_j)] \\
&= \sum_{K,L \in I_m} \left[C_L C_K \mathbb{E}[e_L(X_i)e_K^T(X_j)] + \tilde{C}_L \tilde{C}_K \mathbb{E}[\tilde{e}_L(X_i)\tilde{e}_L^T(X_j)] \right] \\
&\quad + \sum_{r=m+1}^{\infty} \sum_{K,L \in I_r} \left[C_L C_K \mathbb{E}[e_L(X_i)e_K^T(X_j)] + \tilde{C}_L \tilde{C}_K \mathbb{E}[\tilde{e}_L(X_i)\tilde{e}_L^T(X_j)] \right] \\
&= \mathbb{E}[Q_1(i,j)] + \mathbb{E}[Q_2(i,j)], \tag{3.8}
\end{aligned}$$

where

$$Q_1(i,j) = Q_{11}(i,j) + Q_{12}(i,j) \tag{3.9}$$

with

$$\begin{aligned}
Q_{11}(i,j) &= \sum_{K,L \in I_m} \left[C_L C_K e_L(X_i)e_K^T(X_j) \right], \\
Q_{12}(i,j) &= \sum_{K,L \in I_m} \left[\tilde{C}_L \tilde{C}_K \tilde{e}_L(X_i)\tilde{e}_K^T(X_j) \right],
\end{aligned}$$

and

$$Q_2(i,j) = Q_{21}(i,j) + Q_{22}(i,j), \tag{3.10}$$

with

$$\begin{aligned}
Q_{21}(i,j) &= \sum_{r=m+1}^{\infty} \sum_{K,L \in I_r} \left[C_L C_K e_L(X_i)e_K^T(X_j) \right], \\
Q_{22}(i,j) &= \sum_{r=m+1}^{\infty} \sum_{K,L \in I_r} \left[\tilde{C}_L \tilde{C}_K \tilde{e}_L(X_i)\tilde{e}_K^T(X_j) \right].
\end{aligned}$$

Since

$$\mathbb{E} \left[\left(\sum_{i=1}^N G(X_i) \right) \left(\sum_{i=1}^N G^T(X_i) \right) \right] = \mathbb{E} \left[\sum_{i=1}^N \sum_{j=1}^N G(X_i)G^T(X_j) \right],$$

we have

$$\begin{aligned}
& \mathbb{E} \left\| \left[\left(\sum_{i=1}^N G(X_i) \right) \left(\sum_{i=1}^N G^T(X_i) \right) \right] \right\| \leq \\
& \quad \tilde{K} \mathbb{E} \left\| \left[\sum_{i=1}^N \sum_{j=1}^N Q_1(i,j) \right] \right\| + \tilde{K} \mathbb{E} \left\| \left[\sum_{i=1}^N \sum_{j=1}^N Q_2(i,j) \right] \right\|. \tag{3.11}
\end{aligned}$$

By (3.11), in order to show (3.5), we first show that there exists a constant $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$

$$\left\| \mathbb{E} \left[\sum_{i=1}^N \sum_{j=1}^N Q_1(i,j) \right] \right\| \leq \tilde{K} \|N^D \Gamma N^{D*}\|. \tag{3.12}$$

By (3.9), in order to show (3.12), it suffices to show that

$$\left\| \mathbb{E} \left[\sum_{i=1}^N \sum_{j=1}^N Q_{11}(i, j) \right] \right\| \leq \tilde{K} \|N^D \Gamma N^{D*}\|, \quad (3.13)$$

and

$$\left\| \mathbb{E} \left[\sum_{i=1}^N \sum_{j=1}^N Q_{12}(i, j) \right] \right\| \leq \tilde{K} \|N^D \Gamma N^{D*}\|. \quad (3.14)$$

Here, we first assume that $m \neq 0$. Then we have

$$\mathbb{E}[Q_{11}(i, j)] = \mathbb{E} \left[\left(M_{p,q} \right)_{2 \times 2} \right], \quad (3.15)$$

where $M_{1,1} = \sum_{K,L \in I_m} C_L C_K H_{l_1}(X_i^1) H_{l_2}(X_i^2) H_{k_1}(X_j^1) H_{k_2}(X_j^2)$ and $M_{p,q} = 0$ for $q \neq 1$ or $p \neq 1$.

On the other hand, we note that for $M_{1,1}$,

$$\begin{aligned} |\mathbb{E}[M_{1,1}]| &\leq \tilde{K} \sum_{K,L \in I_m} \left| \mathbb{E} \left[C_L^2 \prod_{n=1}^2 H_{l_n}(X_i^n) H_{k_n}(X_j^n) \right] \right| + \\ &\quad + \tilde{K} \sum_{K,L \in I_m} \left| \mathbb{E} \left[C_K^2 \prod_{n=1}^2 H_{l_n}(X_i^n) H_{k_n}(X_j^n) \right] \right| \\ &\leq \tilde{K} \sum_{K,L \in I_m} \mathbb{E} \left[\prod_{n=1}^2 \frac{H_{l_n}(X_i^n) H_{k_n}(X_j^n)}{l_n k_n} \right]. \end{aligned} \quad (3.16)$$

It follows from Sánchez[19] that

$$\begin{aligned} \sum_{K,L \in I_m} \left| \mathbb{E} \left[\prod_{n=1}^2 \frac{H_{l_n}(X_i^n) H_{k_n}(X_j^n)}{l_n k_n} \right] \right| &\leq \frac{1}{m!} \left(\sum_{n,h=1}^2 |\mathbb{E}[X_i^n X_j^h]| \right)^m \\ &\leq \frac{1}{m!} \left(\sum_{n,h=1}^2 |r_{n,h}(i, j)| \right)^m \\ &\leq \tilde{K} \|r(i, j)\|^m. \end{aligned} \quad (3.17)$$

Hence, we can get that

$$\mathbb{E} \left[\sum_{i=1}^N \sum_{j=1}^N Q_{11}(i, j) \right] \leq \tilde{K} \sum_{i=1}^N \sum_{j=1}^N \|r(i, j)\|^m A, \quad (3.18)$$

where

$$A = \begin{bmatrix} 1, 0 \\ 0, 0 \end{bmatrix}. \quad (3.19)$$

Using the same method as the proof of (3.18), we can get that

$$\mathbb{E} \left[\sum_{i=1}^N \sum_{j=1}^N Q_{12}(i, j) \right] \leq \tilde{K} \sum_{i=1}^N \sum_{j=1}^N \|r(i, j)\|^m \tilde{A}, \quad (3.20)$$

where

$$\tilde{A} = \begin{bmatrix} 0, 0 \\ 0, 1 \end{bmatrix}. \quad (3.21)$$

From (3.18) and (3.20), we get that

$$\left\| \mathbb{E} \left[\sum_{i=1}^N \sum_{j=1}^N Q_1(i, j) \right] \right\| \leq \tilde{K} \sum_{i=1}^N \sum_{j=1}^N \|r(i, j)\|^m. \quad (3.22)$$

It follows from (ii) of Definition 3.2 that

$$\left\| \mathbb{E} \left[\sum_{i=1}^N \sum_{j=1}^N Q_{11}(i, j) \right] \right\| \leq \tilde{K} \|N^D \Gamma N^{D*}\|. \quad (3.23)$$

If $m = 0$, we can still use the same method to prove (3.12). From the above arguments, we can get (3.13) holds.

Using the same method as the proof of (3.13), we can prove that

$$\left\| \mathbb{E} \left[\sum_{i=1}^N \sum_{j=1}^N Q_{12}(i, j) \right] \right\| \leq \tilde{K} \|N^D \Gamma N^{D*}\|. \quad (3.24)$$

By (3.11) and (3.24), in order to establish (3.5), it is sufficient to show that as $N \rightarrow \infty$,

$$\left\| \mathbb{E} \left[N^{-D} \sum_{i=1}^N \sum_{j=1}^N Q_2(i, j) N^{-D*} \right] \right\| \rightarrow 0. \quad (3.25)$$

By (3.10), in order to prove (3.5), we need first to look at the components $Q_{21}(i, j)$ and $Q_{22}(i, j)$. We first look at $Q_{21}(i, j)$. In order to simplify the notation, let us define that for $Q \leq N \in \mathbb{N}$,

$$B(N, Q) = \{(i, j) : |i - j| \leq Q, 0 \leq i, j \leq N\},$$

and

$$\tilde{B}(N, Q) = \{(i, j) : |i - j| > Q, 0 \leq i, j \leq N\}.$$

Then, we have

$$\begin{aligned} \sum_{i=1}^N \sum_{j=1}^N \mathbb{E} [Q_{21}(i, j)] &= \sum_{r=m+1}^{\infty} \sum_{(i, j) \in B(N, Q)} \mathbb{E} \left[\left(\sum_{L \in I_r} C_L e_L(X_i) \right) \left(\sum_{K \in I_r} C_K e_K^T(X_i) \right) \right] \\ &\quad + \sum_{r=m+1}^{\infty} \sum_{(i, j) \in \tilde{B}(N, Q)} \mathbb{E} \left[\left(\sum_{L \in I_r} C_L e_L(X_i) \right) \left(\sum_{K \in I_r} C_K e_K^T(X_i) \right) \right] \\ &= \sum_{r=m+1}^{\infty} \mathbb{E} [\tilde{Q}_{21}(N)] + \sum_{r=m+1}^{\infty} \mathbb{E} [\tilde{Q}_{22}(N)], \end{aligned} \quad (3.26)$$

where

$$\mathbb{E}[\tilde{Q}_{21}(N)] = \sum_{(i,j) \in B(N,Q)} \mathbb{E} \left[\left(\sum_{L \in I_r} C_L e_L(X_i) \right) \left(\sum_{K \in I_r} C_K e_K^T(X_i) \right) \right],$$

and

$$\mathbb{E}[\tilde{Q}_{22}(N)] = \sum_{(i,j) \in \tilde{B}(N,Q)} \mathbb{E} \left[\left(\sum_{L \in I_r} C_L e_L(X_i) \right) \left(\sum_{K \in I_r} C_K e_K^T(X_i) \right) \right].$$

Similarly, for $Q_{22}(i, j)$, we also have

$$\sum_{i=1}^N \sum_{j=1}^N \mathbb{E}[Q_{22}(i, j)] = \sum_{r=m+1}^{\infty} \mathbb{E}[\hat{Q}_{21}(N)] + \sum_{r=m+1}^{\infty} \mathbb{E}[\hat{Q}_{22}(N)], \quad (3.27)$$

where

$$\mathbb{E}[\hat{Q}_{21}(N)] = \sum_{(i,j) \in B(N,Q)} \mathbb{E} \left[\left(\sum_{L \in I_r} \tilde{C}_L \tilde{e}_L(X_i) \right) \left(\sum_{K \in I_r} \tilde{C}_K \tilde{e}_K^T(X_i) \right) \right],$$

and

$$\mathbb{E}[\hat{Q}_{22}(N)] = \sum_{(i,j) \in \tilde{B}(N,Q)} \mathbb{E} \left[\left(\sum_{L \in I_r} \tilde{C}_L \tilde{e}_L(X_i) \right) \left(\sum_{K \in I_r} \tilde{C}_K \tilde{e}_K^T(X_i) \right) \right].$$

Therefore, we have

$$\begin{aligned} & \left\| \mathbb{E} \left[N^{-D} \sum_{i=1}^N \sum_{j=1}^N Q_{22}(i, j) N^{-D^*} \right] \right\| \\ &= \left\| \mathbb{E} \left[N^{-D} \sum_{i=1}^N \sum_{j=1}^N [Q_{21}(i, j) + Q_{22}(i, j)] N^{-D^*} \right] \right\| \\ &\leq \left\| \mathbb{E} \left[N^{-D} \sum_{r=m+1}^{\infty} \tilde{Q}_{21}(i, j) N^{-D^*} \right] \right\| + \left\| \mathbb{E} \left[N^{-D} \sum_{r=m+1}^{\infty} \hat{Q}_{21}(i, j) N^{-D^*} \right] \right\| \\ &\quad + \left\| \mathbb{E} \left[N^{-D} \sum_{r=m+1}^{\infty} [\tilde{Q}_{22}(i, j) + \hat{Q}_{22}(i, j)] N^{-D^*} \right] \right\|. \end{aligned} \quad (3.28)$$

Therefore, by (3.28), in order to prove (3.25), we only need to prove that as $N \rightarrow \infty$,

$$\left\| \mathbb{E} \left[N^{-D} \sum_{r=m+1}^{\infty} \tilde{Q}_{21}(i, j) N^{-D^*} \right] \right\| \rightarrow 0, \quad (3.29)$$

$$\left\| \mathbb{E} \left[N^{-D} \sum_{r=m+1}^{\infty} \hat{Q}_{21}(i, j) N^{-D^*} \right] \right\| \rightarrow 0, \quad (3.30)$$

and

$$\left\| \mathbb{E} \left[N^{-D} \sum_{r=m+1}^{\infty} [\tilde{Q}_{22}(i, j) + \hat{Q}_{22}(i, j)] N^{-D^*} \right] \right\| \rightarrow 0. \quad (3.31)$$

Next, we first deal with (3.29). Actually,

$$\mathbb{E}[\tilde{Q}_{21}(N)] = \sum_{(i,j) \in B(N,Q)} \sum_{K,L \in I_r} C_L C_K \mathbb{E}[e_L(X_i) e_K^T(X_j)]. \quad (3.32)$$

Similar to (3.15), we can get that

$$\mathbb{E}\left[\sum_{r=m+1}^{\infty} \tilde{Q}_{21}(N)\right] = \mathbb{E}\left[\sum_{r=m+1}^{\infty} \sum_{(i,j) \in B(N,Q)} (M_{p,q})\right]. \quad (3.33)$$

On the other hand, we have that

$$\begin{aligned} |\mathbb{E}[M_{1,1}]| &= \sum_{L,K \in I_r} \left| \mathbb{E}\left[C_L C_K \prod_{n=1}^2 H_{l_n}(X_i^n) H_{k_n}(X_j^n)\right] \right| \\ &\leq \tilde{K} \sum_{L \in I_r} \left| \mathbb{E}\left[C_L^2 H_{l_1}(X_i^1) H_{l_2}(X_j^2)\right] \right| \\ &\quad + \tilde{K} \sum_{K \in I_r} \left| \mathbb{E}\left[C_K^2 H_{k_1}(X_j^1) H_{k_2}(X_j^2)\right] \right|. \end{aligned} \quad (3.34)$$

It follows from Lemma 5.1 in Sánchez [19] that

$$\mathbb{E}\left[H_{l_1}(X_i^1) H_{l_2}(X_i^2)\right] = l_1! l_2!. \quad (3.35)$$

From the above argument, we get that

$$\begin{aligned} \left\| \mathbb{E}\left[\sum_{r=m+1}^{\infty} \tilde{Q}_{21}(N)\right] \right\| &\leq \tilde{K} \sum_{r=m+1}^{\infty} \sum_{(i,j) \in B(N,Q)} \sum_{L \in I_r} C_L^2 l_1! l_2! \\ &\leq \tilde{K} Q N \sum_{r=m+1}^{\infty} \sum_{l \in I_r} C_L^2 l_1! l_2! \\ &\leq \tilde{K} C_G Q N. \end{aligned} \quad (3.36)$$

On the other hand, by (2.2),

$$\left\| \mathbb{E}\left[N^{-D} \sum_{r=m+1}^{\infty} \tilde{Q}_{21}(N) N^{-D^*}\right] \right\| \leq \tilde{K} N^{-2(\lambda_D - \delta) + 1} \left\| \mathbb{E}\left[\sum_{r=m+1}^{\infty} \tilde{Q}_{21}(N)\right] \right\|, \quad (3.37)$$

with $\delta \in (0, 1)$.

From (3.49) and (3.37), we get that as $N \rightarrow \infty$

$$\left\| \mathbb{E}\left[N^{-D} \sum_{r=m+1}^{\infty} \tilde{Q}_{21}(N) N^{-D^*}\right] \right\| \rightarrow 0, \quad (3.38)$$

since $\lambda_D > \frac{1}{2}$.

Similar to (3.38), as $N \rightarrow \infty$

$$\left\| \mathbb{E} \left[N^{-D} \sum_{r=m+1}^{\infty} \hat{Q}_{21}(N) N^{-D*} \right] \right\| \rightarrow 0. \quad (3.39)$$

Now we deal with (3.31). Similar to (3.15), we have

$$\mathbb{E}[\tilde{Q}_{22}(N)] = \sum_{(i,j) \in \tilde{B}(N,Q)} \left(\mathbb{E}[M_{p,q}] \right)_{2 \times 2}. \quad (3.40)$$

We also note that

$$\begin{aligned} \left| \mathbb{E}[M_{1,1}] \right| &\leq \sum_{K,L \in I_r} \left| \mathbb{E} \left[C_L C_K \prod_{n=1}^2 H_{l_n}(X_i^n) H_{k_n}(X_j^n) \right] \right| \\ &= \sum_{K,L \in I_r} \left\{ \left| \mathbb{E} \left[C_L C_K r! \prod_{n=1}^2 \frac{H_{l_n}(X_i^n) H_{k_n}(X_j^n)}{l_n! k_n!} \right] \right| \right. \\ &\quad \left. \frac{\prod_{n=1}^2 l_n! k_n!}{r!} \right\}. \end{aligned} \quad (3.41)$$

By the Cauchy-Schwartz inequality, we get that

$$\begin{aligned} \sum_{K,L \in I_r} \mathbb{E} \left| \left[C_L C_K \prod_{n=1}^2 H_{l_n}(X_i^n) H_{k_n}(X_j^n) \right] \right| \\ \leq \left\{ \sum_{K,L \in I_r} (C_L^2 C_K^2) \left(\frac{\prod_{n=1}^2 l_n! k_n!}{r!} \right)^2 \right\}^{\frac{1}{2}} \\ \times \left\{ \left(\sum_{K,L \in I_r} r! \mathbb{E} \left[\prod_{n=1}^2 \frac{H_{l_n}(X_i^n) H_{k_n}(X_j^n)}{l_n! k_n!} \right] \right)^2 \right\}^{\frac{1}{2}}. \end{aligned} \quad (3.42)$$

We observe that the last term in (3.42) can be bounded by

$$\begin{aligned} \left\{ \sum_{K,L \in I_r} \left(r! \left| \mathbb{E} \left[\prod_{n=1}^2 \frac{H_{l_n}(X_i^n) H_{k_n}(X_j^n)}{l_n! k_n!} \right] \right| \right)^2 \right\}^{\frac{1}{2}} \\ \leq \tilde{K} \sum_{K,L \in I_r} \left| r! \mathbb{E} \left[\prod_{n=1}^2 \frac{H_{l_n}(X_i^n) H_{k_n}(X_j^n)}{l_n! k_n!} \right] \right|. \end{aligned} \quad (3.43)$$

Moreover, due to Sánchez [19], we obtain that

$$\sum_{K,L \in I_r} \left| r! \mathbb{E} \left[\prod_{n=1}^2 \frac{H_{l_n}(X_i^n) H_{k_n}(X_j^n)}{l_n! k_n!} \right] \right| \leq \left(\sum_{p=1}^2 \sum_{q=1}^2 |r_{p,q}(i,j)| \right)^r. \quad (3.44)$$

On the other hand, we note that

$$\left\{ \sum_{K,L \in I_r} C_K^2 C_L^2 \left(\frac{\prod_{i=1}^n l_n! k_n!}{r!} \right)^2 \right\}^{\frac{1}{2}} \leq \tilde{K} \left(\sum_{L \in I_r} C_L^2 l_1! l_2! \right), \quad (3.45)$$

since $k_1!k_2! \leq r!$. Hence, it follows from (3.40) to (3.45) that

$$\begin{aligned} \sum_{r=m+1}^{\infty} \mathbb{E}[\tilde{Q}_{22}(N)] &\leq \tilde{K} \sum_{r=m+1}^{\infty} \sum_{(i,j) \in \tilde{B}(N,Q)} \left(\sum_{p=1}^2 \sum_{q=1}^2 r_{p,q}(i,j) \right)^r \left(\sum_{L \in I_r} C_L^2 l_1! l_2! \right) A \\ &\leq \tilde{K} \sum_{r=m+1}^{\infty} \sum_{(i,j) \in \tilde{B}(Q,N)} \left(\sum_{p=1}^2 \sum_{q=1}^2 |r_{p,q}(|i-j|)| \right)^r \left(\sum_{L \in I_r} C_L^2 l_1! l_2! \right) A, \end{aligned} \quad (3.46)$$

where A is given by (3.19).

According to (ii) of **Condition** $\mathcal{H}(m, D)$, there exists a constant $\tilde{Q}_0 \in \mathbb{N}$ such that for all $n \geq \tilde{Q}_0$

$$\sum_{p=1}^2 \sum_{q=1}^2 |r_{p,q}(|n|)| \leq \lambda < 1. \quad (3.47)$$

By (3.46) and (3.47), we get

$$\sum_{r=m+1}^{\infty} \mathbb{E}[\tilde{Q}_{22}(N)] \leq \tilde{K} C_G \left(\sum_{p=1}^2 \sum_{q=1}^2 r_{p,q}(|n|) \right)^m A. \quad (3.48)$$

Since for any $m \in \mathbb{N}$,

$$(a+b)^m \leq \tilde{K}(a^m + b^m) \text{ with } a, b > 0,$$

we get from (3.48) that for $N > \tilde{Q}_0$ large enough

$$\mathbb{E}[\tilde{Q}_{22}(N)] \leq \tilde{K} C_G \left(\sum_{p=1}^2 \sum_{q=1}^2 [r_{p,q}(|n|)]^m \right) A. \quad (3.49)$$

By (3.49), we can get that for large enough N

$$\mathbb{E}[\tilde{Q}_{22}(N)] \leq \tilde{K} C_G \|(r_{p,q}(|n|))\|^m A. \quad (3.50)$$

Using the same method as the proof of (3.50), we get that

$$\mathbb{E} \left[\sum_{r=m+1}^{\infty} \hat{Q}_{22}(N) \right] \leq \tilde{K} C_G \|(r_{p,q}(|n|))\|^m \tilde{A}, \quad (3.51)$$

where \tilde{A} is given by (3.21).

It follows from (3.49) and (3.50) that

$$\mathbb{E} \left\| \left[N^{-D} \sum_{r=m+1}^{\infty} [\tilde{Q}_{22}(i,j) + \hat{Q}_{22}(i,j)] N^{-D^*} \right] \right\| \leq \tilde{K} \|r(|i-j|)\|^m. \quad (3.52)$$

Hence, by (ii) of **Condition** $\mathcal{H}(m, D)$ and (3.52), we get (3.31).

From the above arguments, we can get the theorem holds. \square

Remark 3.1 From the proof of Theorem 3.1, we easily get that as $N \rightarrow \infty$,

$$\sum_{i,j=1}^N \left\| \mathbb{E} \left[\sum_{r=m+1}^{\infty} \sum_{K,L \in I_r} N^{-D} \left[C_L C_K e_L(X_i) e_K^T(X_j) N^{-D*} + \tilde{C}_L \tilde{C}_K N^{-D} \tilde{e}_L(X_i) \tilde{e}_K^T(X_j) \right] N^{-D*} \right] \right\| \rightarrow 0. \quad (3.53)$$

3.2. Reduction Theorem

In this subsection, we assume that $G \in \mathbb{G}_m$ and $\{X_i\}$ satisfies **Condition** $\mathcal{H}(m, D)$, and study the weak limit theorem for the process

$$Z_N(t) = N^{-D} \sum_{i=1}^{\lfloor Nt \rfloor} G(X_i). \quad (3.54)$$

For the sake of convenience, we define the following notation.

$$\begin{aligned} Z_{N,m}(t) &= N^{-D} \left[\sum_{i=1}^{\lfloor Nt \rfloor} \sum_{L \in I_m} [C_L e_L(X_i) + \tilde{C}_L \tilde{e}_L(X_i)] \right], \\ &= \sum_{i=1}^{\lfloor Nt \rfloor} \sum_{L \in I_m} C_L N^{-D} e_L(X_i) + \sum_{i=1}^{\lfloor Nt \rfloor} \sum_{L \in I_m} \tilde{C}_L N^{-D} \tilde{e}_L(X_i), \end{aligned} \quad (3.55)$$

and

$$\tilde{Z}_{N,m}(t) = \sum_{i=1}^{\lfloor Nt \rfloor} \sum_{r=m+1}^{\infty} \sum_{L \in I_r} C_L N^{-D} e_L(X_i) + \sum_{i=1}^{\lfloor Nt \rfloor} \sum_{r=m+1}^{\infty} \sum_{L \in I_r} \tilde{C}_L N^{-D} \tilde{e}_L(X_i), \quad (3.56)$$

where

$$C_L = \frac{\mathbb{E}[G^T(X) e_L(X)]}{l_1! l_2!} \text{ and } \tilde{C}_L = \frac{\mathbb{E}[G^T(X) \tilde{e}_L(X)]}{l_1! l_2!}.$$

Before we state our result, we need the following useful lemma.

Lemma 3.1 *If the limits of the finite-dimensional distributions of $(Z_{N,m}(t))$ exist, as $N \rightarrow \infty$, we denote it by $(Z_m(t_1), \dots, Z_m(t_p))$, then as $N \rightarrow \infty$*

$$(Z_N(t_1), \dots, Z_N(t_p)) \xrightarrow{d} \left(\sum_{L \in I_m} (C_L + \tilde{C}_L) Z_m(t_1), \dots, \sum_{L \in I_m} (C_L + \tilde{C}_L) Z_m(t_p) \right), \quad (3.57)$$

where \xrightarrow{d} denotes convergence in distribution.

Proof: In order to simplify the discussion, we only prove the case that $p = 1$. The general case that $p \in \mathbb{N}$ can be done in the same way. According to (3.6), (3.55) and (3.56), in order to prove (3.57), we only need to prove that

$$\tilde{Z}_{N,m}(t) \xrightarrow{d} 0, \quad (3.58)$$

as $N \rightarrow \infty$. To prove (3.58), it is sufficient to prove that $\{\tilde{Z}_{N,m}(t)\}$ converges to zero in probability, i.e., as $N \rightarrow \infty$

$$\mathbb{P}\left\{\|\tilde{Z}_{N,m}(t)\|_2 \geq \epsilon\right\} \rightarrow 0. \quad (3.59)$$

Note that for an \mathbb{R}^d -valued random variable $Q = (Q^1, \dots, Q^d)^T$, $\mathbb{E}[\|Q\|_2^2]$ equals the sum of diagonal entries of the correlation matrix. It follows from the above arguments and (2.2) that

$$\mathbb{E}\left[\|\tilde{Z}_{N,m}(t)\|_2^2\right] \leq \tilde{K} \left\| \mathbb{E}\left[\tilde{Z}_{N,m}(t) \tilde{Z}_{N,m}^T(t)\right] \right\|. \quad (3.60)$$

Hence,

$$\begin{aligned} \mathbb{E}\left[\|\tilde{Z}_{N,m}(t)\|_2^2\right] &\leq \tilde{K} \left\| \mathbb{E}\left[\tilde{Z}_{N,m}(t) \tilde{Z}_{N,m}^T(t)\right] \right\| \\ &\leq \tilde{K} \left\| \mathbb{E}\left[\sum_{i=1}^{\lfloor Nt \rfloor} \sum_{j=1}^{\lfloor Nt \rfloor} \left(\sum_{r=m+1}^{\infty} \sum_{L,K \in I_r} C_L C_K N^{-D} e_L(X_i) e_K^T(X_j) N^{-D^*} \right) \right] \right. \\ &\quad \left. + \mathbb{E}\left[\sum_{i=1}^{\lfloor Nt \rfloor} \sum_{j=1}^{\lfloor Nt \rfloor} \left(\sum_{r=m+1}^{\infty} \sum_{L,K \in I_r} \tilde{C}_L \tilde{C}_K N^{-D} \tilde{e}_L(X_i) \tilde{e}_K^T(X_j) N^{-D^*} \right) \right] \right\| \\ &\leq \tilde{K} \sum_{i,j=1}^N \left\| \mathbb{E}\left[\sum_{r=m+1}^{\infty} \sum_{L,K \in I_r} C_L C_K N^{-D} e_L(X_i) e_K^T(X_j) N^{-D^*} \right] \right. \\ &\quad \left. + \mathbb{E}\left[\sum_{r=m+1}^{\infty} \sum_{L,K \in I_r} \tilde{C}_L \tilde{C}_K N^{-D} \tilde{e}_L(X_i) \tilde{e}_K^T(X_j) N^{-D^*} \right] \right\|, \end{aligned} \quad (3.61)$$

since $t \in (0, 1)$.

Therefore, we get from Remark 3.1 and the Chebyshev-Markov inequality [6, Chap.1] that (3.59) holds. So the lemma holds. \square

By the theorems 2.1, 3.1 and Lemma 3.1, we have

Theorem 3.2 *Let $G \in \mathbb{G}_m$ for some $m \geq 1$, and $\{X_i\}$ satisfies **Condition $\mathcal{H}(m, D)$** . Define $Z_N(t)$ as in (3.54) and $Z_{N,m}(t)$ as in (3.55). If the finite-dimensional distributions of $Z_{N,m}(t)$ converge to $Z_m(t)$, then $Z_N(t)$ converges weakly to the process $\sum_{L \in I_m} (C_L + \tilde{C}_L) Z_m(t)$.*

Proof of Theorem: In order to prove the theorem, it suffices to prove that $\{Z_N(t)\}$ satisfies Theorem 2.1. By Theorem 3.1, we can get the condition (ii') in Remark 2.1 holds. Since $\{X_i\}$ is Gaussian, we can easily get that the condition (iii') in Remark 2.1 holds. Finally, from Lemma 3.1, we get the condition (iv) in Theorem 2.1 holds. From the above argument, we get the theorem holds. \square

4. Limit Theorem for Operator Fractional Brownian Motion

In this section, we present the main result of this paper. We show that when the Hermite Rank m of G is 1, the limiting process turns out to be, up to a multiplicative matrix from

the left, a time reversible operator Brownian motion introduced by Didier and Pipiras [8]. We first recall a useful lemma taken from Didier and Pipiras [8].

Lemma 4.1 *Let D be a linear operator on \mathbb{R}^2 with $0 < \Lambda_D, \lambda_D < 1$. Let $X = \{X(t)\}$ be an OFBM with o.s.s. exponent D . Then X admits the integral representation*

$$X(t) \stackrel{d}{=} \int_{\mathbb{R}} \frac{e^{itx} - 1}{ix} \left(x_+^{-(D-\frac{I}{2})} A + x_-^{-(D-\frac{I}{2})} \bar{A} \right) W(dx) \quad (4.1)$$

for some linear operator A on \mathbb{C}^2 . Here, \bar{A} denotes the complex conjugate and

$$W(x) := W_1(x) + iW_2(x)$$

denotes a complex-valued multivariate Brownian motion such that $W_1(-x) = W_1(x)$ and $W_2(-x) = -W_2(x)$, $W_1(x)$ and $W_2(x)$ are independent, and the induced random measure $W(x)$ satisfies

$$\mathbb{E} \left[W^*(dx) W(dx) \right] = dx,$$

where W^* is the adjoint operator of W .

Inspired by Samorodnitsky and Taqqu [18] [Chap. 7], up to a multiplicative matrix from the left, we can rewrite $\{X(t)\}$ as follows.

$$X(t) \stackrel{d}{=} \int_0^\infty G_1(x, t) W_1(dx) + \int_0^\infty G_2(x, t) W_2(dx), \quad (4.2)$$

where

$$G_1(x, t) = \frac{\sin tx}{x} x^{-(D-\frac{I}{2})} A_1 + \frac{\cos tx - 1}{x} x^{-(D-\frac{I}{2})} A_2,$$

$$G_2(x, t) = \frac{\sin tx}{x} x^{-(D-\frac{I}{2})} A_2 + \frac{1 - \cos tx}{x} x^{-(D-\frac{I}{2})} A_1,$$

and

$$A = A_1 + iA_2.$$

In order to give the main result of this paper, we also need the following lemma, which comes from Dai [5].

Lemma 4.2 *Let $\{Z_i, i = 1, 2, \dots\}$ be a stationary proper mean-zero Gaussian sequence of \mathbb{R}^2 -valued vectors. We define*

$$r(i, j) = \mathbb{E}[Z_i Z_j^T] = \left(r_{k,q}(|i - j|) \right)_{d \times d},$$

Suppose that

$$\sum_{i=1}^N \sum_{j=1}^N r(i, j) \sim \tilde{K} B N^D \Gamma N^{D*} B^*, \text{ as } N \rightarrow \infty, \quad (4.3)$$

where $\Gamma = \mathbb{E}[X(1)X^T(1)]$, $B \in \text{Aut}(\mathbb{R}^2)$ and $\tilde{K} > 0$ is a positive number. Then

$$Q_N(t) = d_N \sum_{i=1}^{\lfloor Nt \rfloor} Z_i, \quad (4.4)$$

with $d_N \sim CN^{-D}B^-$, converges weakly, as $N \rightarrow \infty$ in $\mathcal{D}^2([0,1])$, up to a multiplicative matrix from the left, to the time reversible OFBM X given by (4.2) with $A_2A_1^* = A_1A_2^*$, where $C \in \text{Aut}(\mathbb{R}^2)$.

Now we give the main result of this paper.

Theorem 4.1 Suppose that $G \in \mathbb{G}_1$ and $\{X_i\} \in \mathcal{H}(1, D)$. Define

$$Z_N(t) = N^{-D} \sum_{i=1}^{\lfloor Nt \rfloor} G(Z_i), \quad (4.5)$$

where $\{Z_i\}$ satisfies the conditions in Lemma 4.2. Then $\{Z_N(t)\}$ converges weakly, up to a multiplicative matrix from the left, to the time reversible OFBM $X(t)$, where $X(t)$ is given by (4.2).

Proof of Theorem: It follows from Reduction Theorem 3.2 that, in order to prove the theorem 4.1, it suffices to show $\{Z_{N,m}(t)\}$ converges weakly to $X(t)$, up to a multiplicative matrix from the left. We should point out that when $m = 1$,

$$e_L(X) = (X^1, 0)^T \text{ and } \tilde{e}_L(X) = (0, X^2)^T.$$

Hence,

$$Z_{N,m}(t) = BQ_N(t), \quad (4.6)$$

where B is a multiplicative matrix. Using the same method as the proof of Lemma 4.2, we can get that $\{Z_{N,m}(t)\}$ converges weakly to $X(t)$, up to a multiplicative matrix from left. Therefore, by Theorem 3.2, we can get Theorem 4.1. \square

5. A Final Note

In the sections 3 and 4, we only discuss problems in the two-dimensional space. In fact, following the same steps, we can trivially extend our results to those in the n -dimensional space. Since the generalization is trivial, we skip them here.

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